Dr. J's Guide to Counting

Counting basics

Initial set:

$$\{1, 2, 3, \ldots, n\}.$$

Final set has k elements:

$$\{\underbrace{?, ?, \ldots, ?}_{k \text{ elements}}\} \quad \text{OR} \quad (\underbrace{?, ?, \ldots, ?}_{k \text{ elements}})$$

Two important questions about the final set:

- Does order matter?
  - combination (unordered)
  - permutation (ordered)
- Can elements in the set be reused?
  - with replacement/repetition
  - without replacement/repetition

Counting

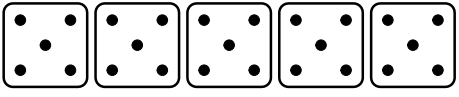
Initial set has n elements and final set has k elements.

Example	Counting method	Ordered	Replacement	Formula
Yahtzee	Combination with replacement	No	Yes	$\binom{n+k-1}{k}$
Lottery	Combination (without replacement)	No	No	$\binom{n}{k}$
Passwords	Permutation with replacement	Yes	Yes	n^k
Cross-over	Permutation (without replacement)	Yes	No	$\frac{n!}{(n-k)!}$

where

- $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$ $(0! = 1)$and
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ and is read "n choose k."

Yahtzee rolls



How many *unique* Yahtzee rolls (5 6-sided dice) are there?

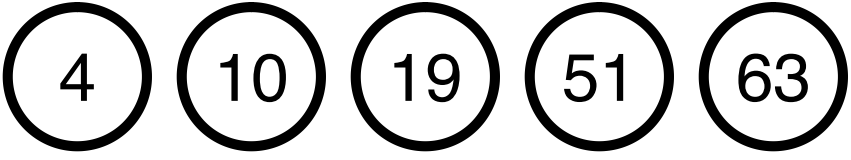
- Initial set: $n = 6$ because each die has 6 sides
- Final set: $k = 5$ because there are 5 dice
- Does order matter? No, 1-1-1-1-2 is the same as 2-1-1-1-1.
- Can elements be reused? Yes, 1-1-1-1-2 is possible.

This is a combination with replacement and there are

$$\binom{6+5-1}{5} = \frac{(6+5-1)!}{5!(6-1)!} = 252$$

unique Yahtzee rolls.

Lottery jackpot draws



How many different lottery picks are there if there are 5 balls with numbers 1 to 66?

- Initial set: $n = 66$ because there are 66 different numbers
- Final set: $k = 5$ because there are 5 balls
- Does order matter? No, 4-10-19-51-63 is the same as 19-51-10-4-63.
- Can elements be reused? No, each ball is drawn at most once.

This is a combination (without replacement) and there are

$$\binom{66}{5} = \frac{66!}{5!(66-5)!} = 8,936,928$$

unique lottery draws.

Passwords

axtkumby

How many passwords of length 8 using lowercase English letters are there?

- Initial set: $n = 26$ because there are 26 English letters.
- Final set: $k = 8$ because the password has 8 digits.
- Does order matter? Yes, axtkumby is not the same as ybmuktxa.
- Can elements be reused? Yes, password policies generally allow reusing digits.

This is a permutation with replacement and there are

$$26^8 = 208,827,064,576$$

unique passwords.

Cross-over clinical trials

How many treatment plans are there for a cross-over clinical trial with 4 treatments where every patients receives 3 different treatments?

- Initial set: $n = 4$ because there are 4 treatments.
- Final set: $k = 3$ because each patient receives 3 treatments.
- Does order matter? Yes, because there are possibly carry-over effects.
- Can elements be reused? No, each treatment a patient receives is different.

This is a permutation (without replacement) and there are

$$\frac{4!}{(4-3)!} = 24$$

treatment plans.

Summary

Counting method	Ordered	Replacement	Formula
Combination with replacement	No	Yes	$\binom{n+k-1}{k}$
Combination (without replacement)	No	No	$\binom{n}{k}$
Permutation with replacement	Yes	Yes	n^k
Permutation (without replacement)	Yes	No	$\frac{n!}{(n-k)!}$

where k is the number of items chosen and n is the number to choose from.